**Ornstein-Uhlenbeck Process**

So the Wiener process forms the backbone of the Ornstein-Uhlenbeck process, which seems to adequately describe all the transfer matrix modeling going on.

**Vector OU Process**

Now let’s say that W(t) drives the evolution of another random variable X(t) via:



where tm+λ = tm + λ(tm+1 – tm) is some time within (tm, tm+1). In normal calculus, it would not matter where tm+λ was located within the interval, but we’ll see it does here. Two special cases stand out: Ito (λ = 0), and Stratonovich (λ = ½). The former is what we implicitly analyzed in the previous section. But the non-zero λ cases result in quite different evolution for X(t) as the a,b and dW terms will now be correlated.

**Integral functions of OU processes**

Let’s consider an integral of a function of a OU process. Let’s have function F(**X**,t):



Now we will now perform a Taylor series expansion of the difference, starting at time tk+λ. As before, we’ll find that the result *does* depend on what λ is. We’ll expand out to first order in dt:



Turns out the last term can be written as a purely deterministic function. To start, we’ll substitute the differential dX equation into this term, keeping only O(dt) terms.



One could demur with respect to the time we evaluate the b’s in the last term, but any choice of t within the interval (tk, tk+1) is equivalent, to our desired order. Now two stochastic functions are considered equal if their average squared difference equals zero (this effectively means the probability of a difference between the two functions is zero). This will imply we can replace the [dWdW - dWdW] term with its average: (1-2λ)DkℓΔtm. To prove this, we’ll take the difference between our present expression and the same expression with the [dWdW – dWdW] term replaced by its average, and analyze the expectation of its square. For concision, we’ll use h(**X**,t) to denote the ∂2F/∂Xi∂Xj·bikbjl term:



Now we can replace h(X(tm),tm) by its supremum over the interval (0,t):



And then using the identity, for independent variables Ai:



we can say:



But now, E[ ] = 0, and var[ ] term will go to zero in the large n limit since (Δtm)2 is of order 1/n2. So we have:



Or in other words:



**Differential functions of OU processes**

We basically have our result above, once the derivative of both sides is taken, but let’s do it independently. So as before, let’s start by considering some function F(X(t)), and its evolution, sans averaging at the moment. To proceed, we’ll expand it out to roughly first order in dt.



Now the dX term is problematic because the X’s in F are evaluated tm, while those in dX are evaluated at tm+λ. Let’s massage everything into the latter time. The ∂F/∂t term can be evaluated at tm+λ w/o cost because the error involved would ultimately be > O(dt) vis a vis dF. So



For the second term, we will make a Taylor expansion about tm+λ, keeping terms out to dt.



One could demur with respect to the time we evaluated *b* in the [Xj(tm) – Xj(tm+λ)] term, but any choice of t within the interval (tm, tm+1) would’ve been equivalent to any other, up to our desired order. Now turns out we can replace the product of W’s with their average in the small dt limit. Heuristically, this is because higher moments of that term are O(dt)2 (but really, this must be proved by placing the derivative within the context of integration). Perhaps this only happens for white noise then?



In dF’s third term, we may switch **X**(tm) → **X**(tm+λ) because the difference doesn’t matter in that term up to our desired order. Then expanding the d**X**’s and keeping only the appropriate order terms we have:



Again we can replace the W product with its average:



Putting it all together we come to:



Finally, dividing both sides by dtm, we arive at our result:



Can the Ito version be obtained heuristically?



But this heuristic will apparently only work for λ = 0. Now let’s consider the product rule:



and so we have:



Again, let’s consider the Ito heuristic:



So using the extended product rule would deliver the Ito version, if D were symmetric. I’d expect it always is. What of the chain rule?



And so we have:



As we can see, the Stratonovich version of the process follows the normal rules of Calculus. This makes it convenient for formal manipulations. Pertinently, one may take an Ito-defined SDE, manipulate it into a Stratonovich-defined SDE, and solve it using the normal rules of Calculus. Once **X**(t) is known (in terms of t and **W**(t)), its moments and probability distribution function may be worked out, in principle.